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Geometrical Interpretation of the Angles α and β in Lambert's Problem

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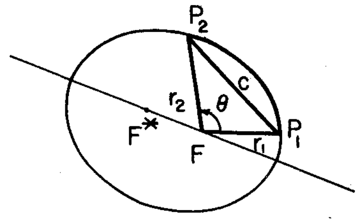

Fig. 1 Geometry of Lambert's problem.

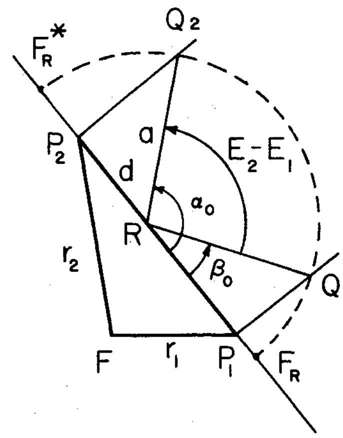

Fig. 2 Interpretation of the angles α_0 and β_0 for an elliptical orbit.

Introduction

LAMBERT'S problem is the two-point boundary-value problem of determining the two-body orbit which connects two given position vectors in a specified flight time. In the classical formulation of the equation for the flight time on an elliptical orbit, due to Lagrange,[1,2] two angles α and β appear, whose values depend on the geometry of the terminal radii and the semimajor axis of the transfer orbit. In this Note, a simple geometrical interpretation of these angles and their analogs for hyperbolic orbits is presented which can be used to geometrically construct these angles in position space. A different interpretation of these angles by Battin[3] in velocity space requires a more complex construction, but allows one to also construct the required velocity vectors at the given terminal radii.

Definitions of α and β

The geometry of Lambert's problem is shown in Fig. 1. The points F and F^* are the focus and vacant focus, respectively, of the transfer ellipse between points P_1 and P_2, and it is assumed that $r_2 \geq r_1$ with no loss of generality. The time of flight $t_F = t_2 - t_1$ between points P_1 and P_2, separated by the chord distance c and the transfer angle θ, is given by[2]

$$\sqrt{\mu}t_F = a^{3/2}[\alpha - \beta - (\sin\alpha - \sin\beta)] \tag{1}$$

where a is the semimajor axis of the elliptical transfer orbit and the angles α and β are defined in terms of the semiperimeter of the Lambert triangle P_1FP_2 of Fig. 1, $s = (r_1 + r_2 + c)/2$ by

$$\sin(\alpha/2) = \sqrt{s/2a} \tag{2}$$

$$\sin(\beta/2) = \sqrt{(s-c)/2a} \tag{3}$$

Equation (1) yields the correct flight time for all possible elliptical paths between P_1 and P_2 for $0 \leq \theta < 2\pi$, which includes the cases for which the transfer angle θ is greater than or less than π and the cases for which the flight time t_F is greater than or less than the time t_m on the minimum-energy ellipse between P_1 and P_2, having semimajor axis $a_m = s/2$. The principal values α_0 and β_0 of the inverse sine functions used to solve Eqs. (2) and (3) are valid in Eq. (1) for the case $\theta \leq \pi$ and $t_F \leq t_m$. In the case that $\theta > \pi$, β in Eq. (1) is equal to $-\beta_0$, and in the case that $t_F > t_m$, α is equal to $2\pi - \alpha_0$. Clearly, $0 \leq \beta_0 \leq \alpha_0 \leq \pi$.

Geometrical Interpretation

The derivation of the geometrical interpretation of the angles α and β is based on two properties of elliptical motion: 1) the flight time satisfies Kepler's equation; and 2) the shape of the transfer orbit can be altered by moving the focus F and the vacant focus F^* without altering the flight time or the angles α and β as long as $r_1 + r_2$ and a remain unchanged in the process. This latter property is discussed in Ref. 2 as an artifice for simplifying the derivation of Eq. (1) for the flight time, and in Ref. 4 as a device for transforming the initial point into an apsidal point (see Fig. 3.5 of Ref. 2 or Fig. 6 of Ref. 4). Using this property, the focus and vacant focus can be moved to the locations F_R and F_R^* shown in Fig. 2, which define the rectilinear elliptical orbit between points P_1 and P_2, which has the same values of $r_1 + r_2$ and a and hence the same flight time, and α and β as the original orbit.

Kepler's equation for the flight time between two points in an elliptical orbit, whose locations are specified by the values of eccentric anomaly E is

$$\sqrt{\mu}t_F = a^{3/2}[E_2 - E_1 - e(\sin E_2 - \sin E_1)] \tag{4}$$

By comparing Eqs. (4) and (1), one can interpret the angles α and β as the values of eccentric anomaly on the rectilinear ellipse ($e = 1$) between P_1 and P_2, having the same values of a and $r_1 + r_2$.

The geometrical interpretation of α and β then follows the usual interpretation of eccentric anomaly. As shown in Fig. 2, one constructs an auxiliary circle of radius a centered at the center R of the rectilinear ellipse. Points Q_1 and Q_2 are the intersections of lines normal to the chord through points P_1 and P_2 with the auxiliary circle. The principal value angles α_0 and β_0 are the angles between the chord line and the auxiliary

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circle radii to points Q_2 and Q_1, as shown in Fig. 2. As mentioned previously, these angles α_0 and β_0 are the correct values of α and β if $\theta \leq \pi$ and $t_F \leq t_m$. If $\theta > \pi$, $\beta = -\beta_0$ and if $t_F > t_m$, $\alpha = 2\pi - \alpha_0$, all of which can be interpreted geometrically in Fig. 2.

The distance d from point P_2 to the rectilinear center R is easily found to be $s-a$ using the fact that $P_1F_R = s-c$ and $P_2F_R^* = 2a-s$. Thus, the center of the auxiliary circle R can be located relative to point P_2 without explicit determination of the location of the rectilinear foci. The geometrical construction is then summarized as follows:

1) Construct a circle of radius a centered at point R a distance $d=s-a$ from P_2. (If $0 \leq d \leq c$, point R lies on the chord; if $d<0$, it lies on the extension of the chord through point P_2; if $d>c$, it lies on the extension of the chord through point P_1.)

2) Construct the two lines normal to the chord through points P_1 and P_2, intersecting the circle at points Q_1 and Q_2.

3) Construct the lines from point R to points Q_1 and Q_2. These lines form the angles β_0 and α_0 with the chord.

Two special cases can be identified in Fig. 2. For the minimum-energy ellipse between P_1 and P_2, $a = a_m = s/2$, points P_2 and F_R^* are coincident, and $\alpha_0 = \pi$. The case in which the rectilinear center R lies at the midpoint of the chord ($d = c/2$) is the symmetric ellipse,[2,4] the ellipse of smallest eccentricity connecting points P_1 and P_2.

It is interesting to note that in Fig. 2 the difference in the values of eccentric anomaly $E_2 - E_1$ on the *original* elliptical path between points P_1 and P_2 can be identified as an angle, using the fact[4,5] that $\alpha - \beta = E_2 - E_1$. The value of $E_2 - E_1$ shown in Fig. 2 is for the case $\theta \leq \pi$, $t_F \leq t_m$. For the other possible cases, $E_2 - E_1$ can also be geometrically interpreted as an angle in Fig. 2.

Hyperbolic Orbits

For hyperbolic orbits, the angles γ and δ are the analogs of α and β and the time-of-flight equation is[2]

$$\sqrt{\mu}\, t_F = a^{3/2}\left[(\sinh\gamma - \sinh\delta) - (\gamma - \delta)\right] \tag{5}$$

where $\sinh(\gamma/2)$ and $\sinh(\delta/2)$ are equal to the right-hand sides of Eqs. (2) and (3), respectively. In Eq. (5), δ is replaced by its negative if $\theta > \pi$.

Kepler's equation for a hyperbolic orbit expressed in terms of hyperbolic-eccentric anomaly H is

$$\sqrt{\mu}\, t_F = a^{3/2}\left[e(\sinh H_2 - \sinh H_1) - (H_2 - H_1)\right] \tag{6}$$

Thus, the angles γ and δ can be interpreted as the values of hyperbolic-eccentric anomaly H_2 and H_1 on the rectilinear hyperbola ($e=1$) between points P_1 and P_2, having the same values of $r_1 + r_2$ and a as the original orbit.

Since the hyperbolic-eccentric anomaly H does not have a geometrical interpretation as an angle, it is convenient to employ the Gudermannian transformation[6,7] from hyperbolic to trigonometric functions $\sinh H = \tan\zeta$, for which $H = \log\tan(\zeta/2 + \pi/4)$. The values of the Gudermannian angles ζ_1 and ζ_2 corresponding to H_1 and H_2 can be interpreted geometrically, using the rectilinear hyperbola in terms of the tangent lines from P_1 and P_2 to the auxiliary circle of radius a centered at R, as shown in Fig. 3. The distance d from R to P_2 is $s+a$ and the angles γ and δ are given by

$$\begin{aligned} \gamma &= \log\tan(\zeta_2/2 + \pi/4) \\ \delta &= \log\tan(\zeta_1/2 + \pi/4) \end{aligned} \tag{7}$$

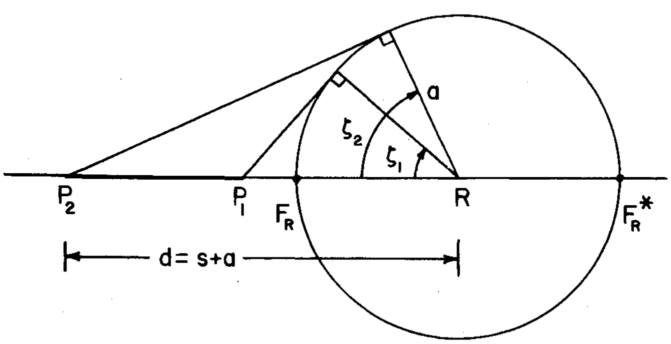

Fig. 3 Interpretation of the angles ζ_1 and ζ_2 for a hyperbolic orbit.

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Relative Motion of Particles in Coplanar Elliptic Orbits

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Introduction

STUDIES of the relative motion of particles in elliptic orbits were primarily prompted by interest in the rendezvous maneuver between an unpowered ferry vehicle and a target satellite in orbit. The relative motion is determined with reference to a coordinate system attached to the target satellite. The purpose of this Note is to expose the need for a simplified solution for coplanar elliptic orbits and to develop and apply such a solution. The results are applicable to any relative motion situation for coplanar orbits such as determining the trajectory of a probe ejected from a space station, rendezvous of orbiting satellites, and the targeting of one space station from another in the same orbital plane.

Brief Survey of Published Solutions

The early solutions, for coplanar orbits and for circular orbits of the target satellite, are obtained from the differential equations of relative motion which are linearized by approximating for small relative displacements. Such solutions are determined by Wheelon,[1] Wolowicz et al.,[2] Spradlin,[3]

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